POPOV INEQUALITY VIA PARAMETER PLANE*

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Abstract

The paper presents a parameter plane interpretation of the Popov inequality which appears in absolute stability analysis of nonlinear systems. The major advantage of the proposed technique over a conventional frequency domain interpretation of the inequality lies in the fact that the influence of system parameters on the absolute stability can be considered by a straightforward procedure. A general computer program is readily available.

Introduction

The problem of absolute stability in a certain class of nonlinear systems was introduced by A.I. Lur'e and V.N. Postnikov [1]. Various solutions of the problem on the basis of the Liapunov direct method were obtained by Lur'e and many others [3]. The solutions are often given for modified and extended versions of the absolute stability problem which is usually referred to as the problem of Lur'e.

As distinct from the other solutions to the Lur'e problem, V.M. Popov [3] expressed the sufficient conditions for absolute stability in terms of the frequence response of the linear part of the system. This resulted in a simple and convenient graphical interpretation common in the linear system analysis. The Popov solution has been further extended to various important problems of nonlinear system analysis. This paper will be based upon the results of V.A. Yakubovich [4,5], Ya.Z. Tsypkin and B.N. Naumov [6,7], and E.I. Jury and A.G. Dewey [8].

In this paper, a solution of the Lur'e problem will be given by reformulating the Popov results in the parameter plane [9-13]. So far, various applications of the parameter plane method utilized the approximate methods of the harmonic linearization (describing function). The obtained results indicated a certain superiority of the parameter plane analysis over the conventional techniques. The method has been applied recently to the problem of Lur'e using a concept of the "complex gain" [14-17]. This concept produced some advantages in comparison to the frequency technique used in the Popov solution. Another approach [18] utilized the properties of the double-real-root loci [9,10] in the parameter plane and offered a promise to achieve new significant results in solving the absolute stability problem in the parameter plane. This approach is generalized here and then applied to various aspects of the absolute stability analysis of nonlinear systems. The main advantage of the proposed approach is that it offers another dimension in the graphical interpretation of the Popov condition. This allows a study of the influence of system parameters on the absolute stability in a class of nonlinear systems. In addition, the use of X_k and Y_k function introduced in [19] allows a convenient application of digital computers to the proposed stability analysis.

The Popov Inequality

In the theorems for absolute stability based upon the frequency domain solution, the Popov inequality is essential. The form of the Popov inequality varies with the problem specifications involved. Several of the basic forms will be reviewed.

In a free dynamic system, the exponential absolute stability is based upon the inequality [4,6,7]

$$\pi(k,q,\sigma,\omega) \equiv \frac{1}{k} + \mathrm{Re}(1+jq\omega)G(\sigma+j\omega) > 0 \qquad (1)$$

for all real $\omega \geq 0$

where $G(s)$ is the transfer function of the linear part of the system,

$$G(s) = \frac{C(s)}{B(s)} = \frac{\sum_{k=0}^{m} c_k s^k}{\sum_{k=0}^{n} b_k s^k}, \quad n \geq m \qquad (2)$$

and $s = \sigma + j\omega$.

Exponential stability of the solution in a forced system depends on the verification of the inequality [4]

$$\pi(k,\sigma,\omega) \equiv \frac{1}{k} + \mathrm{Re}\, G(\sigma+j\omega) > 0 \qquad (3)$$

for all real $\omega \geq 0$

The original Popov criterion [3] for free dynamic systems which is related to inequality 1 when $\sigma = 0$ can be improved by using the inequality [5,8]

$$\pi(k,p,q,\omega) \equiv \frac{1}{k} + \mathrm{Re}(1 + p\omega^2 + jq\omega)G(j\omega) > 0, \qquad (4)$$

for all real $\omega \geq 0$

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In general, to achieve the absolute stability it is necessary to choose parameters k, p, q, or a system parameter in the linear part of the system so that the corresponding inequality is satisfied. This choice can be advantageously made in the parameter plane as is shown in the following section.

Parameter Plane Analysis

To obtain the geometric interpretation of the Popov inequality in the parameter plane, it is necessary first to observe that the above inequalities can be represented by

$$\pi(\alpha,\beta,\sigma,\omega) > 0, \quad \text{for all real } \omega \geq 0 \tag{5}$$

where α and β represents a pair of variables among k, p, q, or some parameter in the transfer function $G(s)$ of the linear part of the system.

If σ is assumed as constant, inequality 5 may be rewritten as

$$\pi(\alpha,\beta,\omega) > 0, \quad \text{for all real } \omega \geq 0 \tag{6}$$

Case (6) will be considered first.

Taking equality in (6), one obtains equation

$$\pi(\alpha,\beta,\omega) = 0 \tag{7}$$

which for a specific value of ω determines a curve C in the parameter $\alpha\beta$-plane. This curve (in most cases a straight line) divides the $\alpha\beta$-plane into regions $\pi > 0$ and $\pi < 0$. Then, to determine the region $P(\pi > 0)$ for all real $\omega \geq 0$, it is necessary to construct the envelope Γ of all the curves C obtained for various values of ω. Existence of an envelope in this case is assured by the fact that the transfer function $G(s)$ is a rational function of the variable s. [20]

Assuming that an envelope exists, let $M(\alpha,\beta)$ be the point of tangency of E with that one of the curves C which corresponds to a certain value ω. The quantities α and β are unknown functions of

$$\alpha = \phi(\omega)$$
$$\beta = \psi(\omega) \tag{8}$$

which satisfy equation 7. In order to determine these functions, it is necessary to use the fact that the tangents to the two curves E and C coincide for all values of ω. Then, a necessary condition for tangency is

$$\frac{\frac{d\alpha}{d\omega}}{\delta\alpha} = \frac{\frac{d\beta}{d\omega}}{\delta\beta} \tag{9}$$

where $d\alpha/d\omega$ and $d\beta/d\omega$ are the derivatives of the unknown functions ϕ and ψ of equations 8, and $\delta\alpha$ and $\delta\beta$ are two quantities proportional to the direction cosines of the tangent to the curve C.

Since ω in equation 7 has a constant value for the particular curve C considered, we have

$$\frac{\partial\pi}{\partial\alpha}\,\delta\alpha + \frac{\partial\pi}{\partial\beta}\,\delta\beta = 0 \tag{10}$$

which determines the tangent to C. Again, the two unknown functions $\alpha = \phi(\omega)$, $\beta = \psi(\omega)$ satisfy the equation 7 also, where ω is now the independent variable. Therefore,

$$\frac{\partial\pi}{\partial\alpha}\frac{d\alpha}{d\omega} + \frac{\partial\pi}{\partial\beta}\frac{d\beta}{d\omega} + \frac{\partial\pi}{\partial\omega} = 0 \tag{11}$$

or, combining equations 7, 9, 10, and 11,

$$\pi(\alpha,\beta,\omega) = 0, \quad \frac{\partial\pi}{\partial\omega} = 0 \tag{12}$$

the two unknown functions $\alpha = \phi(\omega)$, $\beta = \psi(\omega)$ are solutions of equations 12. Hence, the equation of the envelope, in case an envelope exists, is to be found by eliminating the variable ω from equations 12.

Let $E(\alpha,\beta) = 0$ be the equation obtained by eliminating ω from (12), and let us try to determine whether or not this equation represents an envelope of the given curves. Let C_o be the particular curve which corresponds to a value ω_o, and let $M_o(\alpha_o,\beta_o)$ be the point intersections of the two curves

$$\pi(\alpha,\beta,\omega_o) = 0, \quad \frac{\partial\pi}{\partial\omega_o} = 0 \tag{13}$$

For $\omega = \omega_o$, from equations 8 one has $\alpha_o = \phi(\omega_o)$ and $\beta_o = \psi(\omega_o)$ so that

$$\frac{\partial\pi}{\partial\alpha_o}\left(\frac{d\alpha}{d\omega}\right)_o + \frac{\partial\pi}{\partial\beta_o}\left(\frac{d\beta}{d\omega}\right)_o = 0 \tag{14}$$

This equation taken in connection with equation 10 shows that the tangent to the curve C_o coincides with the tangent to the curve described by the point $M(\alpha,\beta)$, at least unless $\partial\pi/\partial\alpha$ and $\partial\pi/\partial\beta$ are both zero, that is, unless the point $M_o(\alpha_o,\beta_o)$ is a singular point for the curve C_o. In general, it follows that the curve $E(\alpha,\beta) = 0$ is composed of two analytically distinct parts, one of which is the true envelope, while the other is the locus of the singular points.

In the cases of the Popov inequality mentioned above, $G(s)$ is a rational function of s, and $\pi(\alpha,\beta,\omega) = 0$ is a polynomial of degree 2n in ω. For any specific pair of values of α and β the equation

$$\pi(\alpha,\beta,\omega) \equiv \sum_{k=0}^{2n} a_k\omega^k = 0 \tag{15}$$

where $a_k = a_k(\alpha_o,\beta_o)$ will have generally 2n distinct roots. Through the corresponding point $M(\alpha,\beta)$ pass, in general, 2n different curves of the given family. But if the point M lies on the curve $E(\alpha,\beta) = 0$, the equations 12 are satisfied simultaneously, and equation 15 for the chosen specific values of α and β has a double root. If only real values of ω are considered as it is required in the Popov inequality, $E(\alpha,\beta) = 0$

represents the double-real-root-loci [9,10] of equation 15.

This result can be readily extended to the cases when σ is not a constant in the Popov inequality. Then it may become necessary to find the envelope of a family of curves

$$\pi(\alpha,\beta,\sigma,\omega) = 0 \tag{1}$$

whose equation involves two variables σ and ω which themselves satisfy a relation of the form*

$$\theta(\sigma,\omega) = 0 \tag{16}$$

This case can be treated in essentially the same way as the previous one. By the rule obtained above, we should join with the given equation the equation obtained by equating to zero the derivative of its left hand member with respect to σ. Thus,

$$\frac{\partial\pi}{\partial\sigma} + \frac{\partial\pi}{\partial\omega}\frac{d\omega}{d\sigma} = 0 \tag{17}$$

where ω is thought of as a function of σ defined by (16). But from (16) we have also

$$\frac{\partial\theta}{\partial\sigma} + \frac{\partial\theta}{\partial\omega}\frac{\partial\omega}{\partial\sigma} = 0 \tag{18}$$

which together with equations 1 and 16 determine the required envelope. The parameters σ and ω may be eliminated between these three equations if desired.

Computer Application

It is of interest to obtain the above relations in a convenient form for computer programming. Consider, for example, inequality 1, where the function $G = G(s)$ is a rational function of $s = \sigma + j\omega$. From (2), one has

$$G = \frac{C_1 + jC_2}{B_1 + jB_2} \tag{19}$$

where

$$B_1 = \sum_{k=0}^{n} b_k X_k \,, \quad B_2 = \sum_{k=0}^{n} b_k Y_k$$

$$C_1 = \sum_{k=0}^{m} c_k X_k \,, \quad C_2 = \sum_{k=0}^{m} c_k Y_k \tag{20}$$

and $X_k = X_k(\sigma,\omega)$, $Y_k = Y_k(\sigma,\omega)$ are functions defined in reference 20 as

$$X_k = \sum_{\nu=0}^{k} (-1)^{\nu}\binom{k}{2\nu}\sigma^{k-2\nu}\omega^{2\nu} \tag{21}$$

$$Y_k = \sum_{\mu=1}^{k} (-1)^{\mu-1}\binom{k}{2\mu-1}\sigma^{k-2\mu+1}\omega^{2\mu-1}$$

Functions X_k and Y_k can readily be calculated from the recurrence formulas

$$X_{k+1} - 2X_1X_k + (X_1^2 + Y_1^2)X_{k-1} = 0$$

$$Y_{k+1} - 2X_1Y_k + (X_1^2 + Y_1^2)Y_{k-1} = 0 \tag{22}$$

where $X_0 \equiv 1$, $X_1 \equiv \sigma$, $Y_0 \equiv 0$, $Y_1 \equiv \omega$.

If $1/k$ and q are considered in (1) as the parameters α and β, respectively, for σ = const. equations 12 become

$$U\alpha + V\beta + W = 0$$

$$U'\alpha + V'\beta + W' = 0 \tag{23}$$

where

$$U = B_1^2 + B_2^2$$

$$V = \omega(B_2C_1 - B_1C_2) \tag{24}$$

$$W = B_1C_1 + B_2C_2$$

and $U' = dU/d\omega$, $V' = dV/d\omega$, $W' = dW/d\omega$. These derivatives can be computed readily by the use of the recurrence relationship

$$X'_{k+1} - 2X_1X'_k + (X_1^2 + Y_1^2)X'_{k-1} + 2Y_1X_{k-1} = 0$$

$$Y'_{k+1} - 2X_1Y'_k + (X_1^2 + Y_1^2)Y'_{k-1} + 2Y_1Y_{k-1} = 0 \tag{25}$$

where X'_k and Y'_k are derivatives of X_k and Y_k with respect to ω. Equations 25 are obtained by differentiating recurrence formulas 22 with respect to ω.

Now, a general computer program can be easily obtained for plotting the envelope $E(\alpha,\beta) = 0$ by simply solving two linear equations 22 for α and β every time the value of ω is changed. On the basis of the above recurrence relationships, the program can be written for highest order of $G(s)$ expected in the applications. Then, the program for specific situations needs only the starting numerical values of the coefficients a_k, b_k, and σ.

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* This case occurs, for example, when nonlinear sampled-data systems are considered, where the equation 16 becomes $\sigma^2 + \omega^2 = \rho^2$, $0 < \rho < 1$. It is also related to the case of nonlinear continuous systems in which a damping coefficient $\zeta = -\sigma/\omega$ is specified.

If the inequality 3 is considered with

$$G(s) = \frac{C(s) + \beta D(s)}{B(s)} \tag{26}$$

where β is an adjustable system parameter, it is necessary to solve equations 23 in which

$$\begin{aligned} U &= B_1^2 + B_2^2 \\ V &= B_1 D_1 + B_2 D_2 \\ W &= B_1 C_1 + B_2 C_2 \end{aligned} \tag{27}$$

and

$$D_1 = \sum_{k=0}^{m} d_k X_k \, , \quad D_2 = \sum_{k=0}^{m} d_k Y_k \quad . \tag{28}$$

Example

Consider the inequality 3 with

$$G(s) = \frac{s^2 + \beta}{s^3 + 2s^2 + s + 1} \tag{29}$$

and $\alpha = 1/k$, $\sigma = 0$. It is necessary to choose the parameter β so that α is at minimum.

The parameter plane diagram is plotted on Fig. 1 and the region $F(\pi>o)$ is determined following the envelope E from a to d.

This example was investigated in reference 21 for $\beta = 0$. Clearly, the minimum of α is obtained for $\beta = 0.5$ and it lies on the β axis which allows k to become infinite (α=0).

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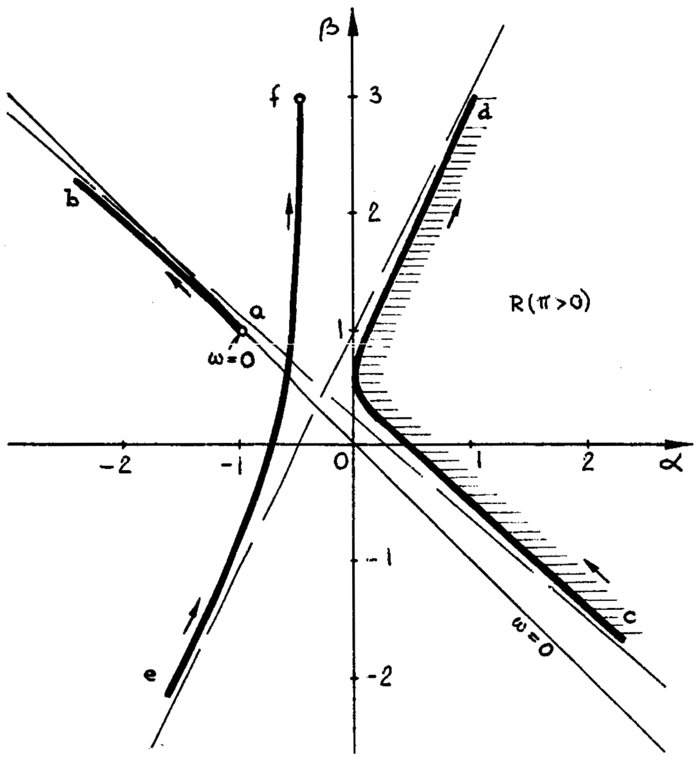

Fig. 1. Parameter plane diagram